On the Quadratic Convergence of the Special Cyclic Jacobi Method

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1. Introduction and Summary

The special cyclic Jacobi method for computing the eigenvalues and eigenvectors of a real symmetric matrix annihilates the off-diagonal elements of the matrix successively, in the natural order, by rows or columns. The convergence of this method was proved by Forsythe and Henrici [1], on the assumption that the rotation-angles $\theta_k$ ($k=0, 1, 2, \ldots$) satisfy the inequality

$$-\frac{1}{3} \pi < \theta^- \leq \theta_k \leq \theta^+ < \frac{1}{3} \pi.$$  

Henrici [2], furthermore, proved the quadratic nature of this convergence for cases where the eigenvalues are distinct, and Wilkinson [4] derived sharp bounds for such cases.

We propose to demonstrate the quadratic nature of the convergence without the assumption of distinct eigenvalues. For our proof, however, we do require that the diagonal elements which converge to the same eigenvalue occupy successive positions on the diagonal. Although this may seem to be a rather stiff requirement, it should be remembered that, once the diagonal elements have the required ordering, they hold it, because, as proved in [1], no diagonal element can change its affiliation if condition (1), as defined below, is satisfied.

2. Proof of the Special Row-Cyclic Method

We assume that the reader is well acquainted with the special cyclic Jacobi method [7]. We have used the notation of Wilkinson [4].

Suppose that after $r$ rotations the matrix $A^{(r)}$ satisfies the condition

$$S^{(r)} \leq \frac{\delta}{4}$$  

where

$$S^{(r)} = \sqrt{\sum_{p < q} (a^{(r)}_{pq})^2}$$  

and

$$2\delta \leq \min_{i \neq j} |\lambda_i - \lambda_j|.$$  

Then we have for two diagonal elements $a_{ii}$ and $a_{ff}$ which converge to unequal eigenvalues $\lambda_{\mu}$ and $\lambda_{\mu'}$, respectively,

$$|a_{ii}^{(r)} - a_{ff}^{(r)}| \geq |\lambda_{\mu} - \lambda_{\mu'}| - |a_{ii}^{(0)} - \lambda_{\mu}| - |a_{ff}^{(0)} - \lambda_{\mu'}|$$

$$> 2\delta - 2\sqrt{\sum_{p \neq q} (a^{(0)}_{pq})^2}$$

$$> 2\delta - 4S^{(0)} > \delta.$$
The quantity $S^{(r)}$ decreases after each subsequent rotation, so that (1) and (4) keep true for the further iteration process.

Since we consider the case that the diagonal elements, which converge to the same eigenvalue, occupy successive positions on the diagonal, we may assume for convenience, without loss of generality, that only the eigenvalue $\lambda_1$ is not simple and that the diagonal elements $a_{11}, a_{22}, \ldots, a_{nn}$ converge to $\lambda_1$.

We first estimate for any $k > r$ the quantity

$$S_k^{(r)} = \sqrt{\sum_{1 \leq p < q \leq m} (a_{pq}^{(k)})^2}$$

by means of a method used by Wilkinson [5]. To this end we partition $A^{(k)}$ in the form

$$A^{(k)} = \begin{bmatrix} A_1^{(k)} & A_2^{(k)} \\ A_2^{(k)T} & A_3^{(k)} \end{bmatrix}$$

such that only the diagonal elements of $A_1^{(k)}$ converge to $\lambda_1$. We then have, if $n$ is the order of the matrix $A^{(k)}$,

$$A_1^{(k)} - \lambda_1 I_n = (A_2^{(k)})^T (A_3^{(k)} - \lambda_1 I_{n-m})^{-1} A_2^{(k)}$$

which gives, if we denote the eigenvalues of $A_3^{(k)}$ by $\lambda'_j$,

$$(S_1^{(k)})^2 \leq \left( \|A_1^{(k)} - \lambda_1 I_n\|_E \right)^2 \leq \frac{\left( \|A_1^{(k)}\|_E \right)^4}{\min_{j \neq 1} |\lambda_1 - \lambda'_j|^2} \leq \frac{4 (S^{(r)})^4}{9 \delta^2} \leq \frac{4 (S^{(r)})^4}{9 \delta^2}$$

since $|\lambda_1 - \lambda'_j| \geq |\lambda_1 - \lambda_j| - |\lambda_j - \lambda'_j| \geq 2 \delta - \frac{\delta}{2} = \frac{3}{2} \delta$ as follows from (1) and (3). ($\|X\|_E$ denotes the euclidean norm of $X$.) This estimate is essential for proving the quadratic convergence of the special cyclic Jacobi method.

Now if during the $k$-th rotation the element $a_{pq}^{(k-1)}$, which is not in $A_1$, is annihilated, then, since the rotation-angle $\theta_k$ is chosen such that $|\theta_k| \leq \pi/4$, we obtain by using (4)

$$|\sin \theta_k| \leq \frac{1}{2} \left| \tan 2\theta_k \right| \leq \frac{|a_{pq}^{(k-1)}|}{|a_{pq}^{(k-2)} - a_{pq}^{(k-1)}|} \leq \frac{|a_{pq}^{(k-1)}|}{\delta}$$

from which, using $(S^{(k-1)})^2 - (S^{(k)})^2 = (a_{pq}^{(k-1)})^2$, we derive the inequality

$$\Sigma' \sin^2 \theta_k \leq \frac{(S^{(r)})^2}{\delta^4}$$

where $\Sigma'$ denotes that we include in the sum only rotations of elements outside $A_1$.

After annihilating the elements of the first row we find, in the same way as in [4],

$$\sum_{m+1}^{n} (a_{i+i+n-1}^{(r+n-1)})^2 \leq (S^{(r)})^2 (\sin^2 \theta_{r+m} + \cdots + \sin^2 \theta_{r+n-1}).$$

Since during subsequent rotations of that cycle the sum of the squares of these elements is unaltered, we obtain on similar lines to relation (3.8) of [4],
by using (5) and (6),

\[(S^{(r+N)})^2 \leq \sum_{p<q \leq m} (a_{p+q}^{(r+N)})^2 + \sum_{p<q, q>m} (a_{p+q}^{(r+N)})^2 \]

\[\leq \frac{1}{\delta^2} (S^{(r+N)})^2 + (S^{(r)})^2 (\sum \sin^2 \theta_k) \]

\[\leq \frac{2 (S^{(r)})^4}{\delta^2} + \frac{(S^{(r)})^4}{\delta^2} \]

which means quadratic convergence.

By an obvious generalization, if \( l \) is the total number of multiple eigenvalues, we have

\[(S^{(r+N)})^2 \leq \left(1 + \frac{2l}{\delta} \right) (S^{(r)})^4. \quad (7)\]

It is possible to decrease the factor \( 1 + \frac{2l}{\delta} \) as follows. If \( a_{\nu_1 \nu_1}, \ldots, a_{\nu_l+\mu_l, \nu_l+\mu_l} \) converge to \( \lambda_i \) (\( 1 \leq i \leq l \)) then we find

\[(S^{(k)})^2 \leq \left(\|A^{(r)}_k - \lambda_i I_m\|_E\right)^2 \leq \left[ \sum_{\nu, \mu} \left( a_{\nu+\mu, \nu+\mu}^{(k)} \right)^2 \right] \leq \frac{8 (S^{(k)})^2}{9 \delta^2} \leq \sum_{\nu, \mu} \left( a_{\nu+\mu, \nu+\mu}^{(k)} \right)^2. \]

Since

\[\sum_{i=1}^l \left[ \sum_{\nu, \mu} \left( a_{\nu+\mu, \nu+\mu}^{(k)} \right)^2 \right] \leq \sum_{\nu, \mu} \left( a_{\nu+\mu, \nu+\mu}^{(k)} \right)^2 = 2 (S^{(k)})^2 \]

we obtain

\[\sum_{i=1}^l (S^{(k)})^2 \leq \frac{16 (S^{(k)})^4}{9 \delta^2} \lesssim \frac{16 (S^{(r)})^4}{9 \delta^2}. \]

In consequence, (7) can be replaced by

\[(S^{(r+N)})^2 \leq \frac{1}{\delta} \sum_{i=1}^l (S^{(r)})^2 + (S^{(r)})^2 (\sum \sin^2 \theta_k) \]

\[\leq \frac{17 (S^{(r)})^4}{9 \delta^2} \]

so that finally

\[S^{(r+N)} \leq \sqrt{\frac{17}{9} \frac{(S^{(r)})^2}{\delta}}. \]

This is worse by a factor of \( \sqrt{\frac{17}{9}} \) than the estimate in \([4]\) for matrices with distinct eigenvalues.

We may add that the proof of the special cyclic column-cyclic method can be given in a similar way to that used here.

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References


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